SPhT 94/156
hep-th/9412202

Conformal, Integrable and Topological Theories, Graphs and Coxeter Groups

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I review three different problems occuring in two dimensional field theory: 1) classification of conformal field theories; 2) construction of lattice integrable realizations of the latter; 3) solutions to the WDVV equations of topological field theories. I show that a structure of Coxeter group is hidden behind these three related problems.

12/94 Talk delivered at the International Conference of Mathematical Physics, Paris 18-23 July 1994

0. Introduction

In this talk, I want to discuss connections between several topics of current interest in mathematical physics, namely certain classes of conformal field theories, integrable lattice models and topological field theories. This list is not limitative, and might have included more items, such as integrable hierarchies and matrix models. The relations between these subjects are not very tight, certainly not one-to-one, but they look strong enough to teach us about one starting from the other. The issue is first the classification programme of these various kinds of theories. But maybe even more important is to understand the principles that are beyond these classifications, and the algebraic or geometric features common to these problems. As may be anticipated from the title of this contribution, I believe that such a common feature is to be found in the existence of a graph (or a collection of graphs); on the one hand, this graph, through its adjacency matrix and its eigenspectrum, codes some important data on the conformal theory or allows the construction of an integrable model; on the other, it also codes the geometry of a root system and hence of a reflection (or Coxeter) group, that appears naturally in the study of topological theories, as shown recently by Dubrovin. Although I do not have yet a complete and consistent picture, the evidence seems compelling enough. I have tried to make this contribution readable by non specialists. Thus the first three sections will be made of material that can hardly be called original, namely a lightning review of the three kinds of problems that we are interested in and the discussion of the results in the cases of theories based on $SU(2)$ and of $SU(3)$. Only in sect. 4 shall I come to some recent results.

1. A short review of the main protagonists

1.1. Conformal Field Theories

Rational conformal field theories (cft) are constructed out of representations of the product $\mathcal{A}\otimes\mathcal{A}$ of two copies of an algebra $\mathcal{A}$ [1–2]. The algebra $\mathcal{A}$ is the maximally extended "chiral" algebra, generated by the product of holomorphic fields in the theory. It contains the Virasoro algebra as a subalgebra (or a subalgebra of its enveloping algebra); examples are provided by the Virasoro algebra itself or its supersymmetric extensions, by affine (or "current") algebras $\widehat{\mathfrak{g}}$, by the so-called W algebras, ... A central role is played by the theories with an affine algebra $\widehat{\mathfrak{g}}$, as it seems likely that all the others may be obtained

from them by the so-called coset construction [3]. The data that specify such a cft (for a given $\mathcal{A}$) are

$$\{k\,,\ (\lambda_i, \lambda_{\bar\imath})\,,\ C_{IJK}\} \tag{1.1}$$

where k is a generic notation for the central charge(s) of $\mathcal{A}$; $(\lambda_i, \lambda_{\bar\imath})$ describe the set of irreducible representations of $\mathcal{A}\otimes\mathcal{A}$ that appear in the decomposition of the Hilbert space of the theory

$$\mathcal{H} = \oplus_{I=(i,\bar\imath)} V_{\lambda_i,k} \otimes V_{\lambda_{\bar\imath},k} \tag{1.2}$$

where V_λ is an irreducible representation of $\mathcal{A}$ of highest weight (h.w.) λ. Here and in the following, a capital like I denotes a pair of indices $(i,\bar\imath)$. The vacuum state plays a special role in the theory: it appears in a representation denoted $\lambda = \bar\lambda = \mathbf{1}$.

If $\mathcal{A} \neq$ Vir such a V_λ splits into a finite or infinite sum of irreducible h.w. representations of the Virasoro algebra : $V_\lambda = \oplus V^{(\mathrm{Vir})}_{h_i}$, and the h_i, eigenvalues of the dilatation generator L_0, are called conformal weights. Finally, in (1.1), C_{IJK} are the structure constants of the operator product algebra (OPA) of the "primary" fields (highest weight representations of $\mathcal{A}\otimes\mathcal{A}$)

$$\Phi_I(z,\bar z)\Phi_J(w,\bar w) = \sum_K C_{IJK}(z-w)^{h_k-h_i-h_j}(\bar z-\bar w)^{h_{\bar k}-h_{\bar\imath}-h_{\bar\jmath}}\{\Phi_K(w,\bar w) + \mathcal{O}(|z-w|)\} \tag{1.3}$$

with $h_i, h_{\bar\imath}$ the conformal weights of Φ_I. In the simplest case of $\mathcal{A} =$ Vir, these data reduce to $\{c, (h_i, h_{\bar\imath}), C_{IJK}\}$, c the Virasoro central charge.

These data are strongly constrained by consistency conditions in the matching between the "left" (holomorphic) and "right" (antiholomorphic) components of the theory. This is exploited in two parallel procedures that deal with the 0-point function in genus 1 and with the 4-point function in genus 0 and that determine respectively the set $(\lambda_i, \lambda_{\bar\imath})$ and the C_{IJK}.

Modular invariance of the partition function on a torus

One considers first the genus 1 0-point function of the cft on the torus $\mathbb{C}/\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. It may be written as

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}}\, q^{L_0 - c/24}\, \bar q^{\bar L_0 - c/24} \tag{1.4}$$

where $L_0, \bar L_0$ are the Virasoro (dilatation) generators, $\tau = \omega_2/\omega_1$ is the modular ratio of the torus and $q = \exp 2i\pi\tau$ is its nome. According to (1.2), this may be rewritten as [4]

$$Z = \sum_{(\lambda_i,\lambda_{\bar\imath})} N_{\lambda_i,\lambda_{\bar\imath}} \chi_{\lambda_i}(q) \chi_{\lambda_{\bar\imath}}(\bar q). \tag{1.5}$$

$N_{\lambda_i,\lambda_{\bar{\imath}}}$ is the multiplicity of occurrence of $\lambda_i, \lambda_{\bar{\imath}}$ in (1.2), hence a non negative integer and

$$\chi_{\lambda}(q) = \mathrm{tr}_{V_{\lambda,k}} q^{L_0 - c/24} \tag{1.6}$$

is the character of the (λ, k) representation of the $\mathcal{A}$ algebra, a generating function of the number of independent states of that representation graded by the eigenvalues of $L_0 - c/24$. The unicity of the vacuum is expressed by

$$N_{\mathbf{1}\,\mathbf{1}} = 1 \ . \tag{1.7}$$

The partition function Z is required to be intrinsically attached to the torus, thus be independent of $SL(2,\mathbb{Z})$ redefinitions of its periods [4–5]. In other words, it must be a modular invariant function of τ

$$Z(\tau) = Z\left(\frac{a\tau+b}{c\tau+d}\right), \qquad a,b,c,d \in \mathbb{Z},\ ad-bc=1 \ . \tag{1.8}$$

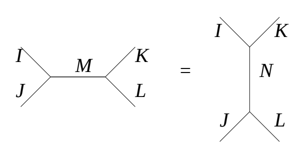
This leads to the following

Problem 1 : *For a given algebra $\mathcal{A}$, classify all sesquilinear forms in the characters (1.5) with non negative integer coefficients and $N_{\mathbf{1}\,\mathbf{1}} = 1$.*
Solution to this problem determines the representation content (1.2).

Monodromy invariance of the 4-point function
After use of a Möbius transformation mapping three of the arguments to 0, 1 and ∞, a correlation function of 4 primary fields may be expressed as [1]

$$\langle \Phi_I(\infty)\Phi_J(1)\Phi_K(z,\bar z)\Phi_L(0)\rangle = \sum_{M=(m,\bar m)} C_{IJM} C_{KLM} \mathcal{J}_m(z)\mathcal{J}_{\bar m}(\bar z) \tag{1.9a}$$

$$= \sum_{N=(n,\bar n)} C_{IKN} C_{JLN} \mathcal{K}_n(1-z)\mathcal{K}_{\bar n}(1-\bar z) \ . \tag{1.9b}$$

written symbolically as 

The so-called conformal blocks $\mathcal{J}_m(z)$ behave as $\mathcal{J}_m(z) \approx z^{h_m - h_k - h_l}$ as $z \to 0$ and likewise for $\mathcal{K}(1-z)$ close to $z=1$. They have non trivial monodromy as z circles around 0, 1 or ∞. Locality requires that the function be single-valued, hence that the combination (1.9a) be monodromy invariant, or equivalently, that (1.9a) and (1.9b) be consistent. Hence the

Problem 2 : *Find all the monodromy invariant combinations of conformal blocks* (1.9a)*.*
This determines the set of structure constants C_{IJK}.

1.2. Integrable height lattice models

Height or *solid-on-solid* models are statistical mechanics models originally designed to describe a fluctuating surface or interface with no overhang : to each site i of a two-dimensional lattice is attached an integer a_i, describing the height of the surface above a reference plane. In the so-called restricted solid-on-solid models (RSOS), additional constraints are put on the range of a : $1 \le a \le n$, and on its fluctuations from site to site, typically $|a_i - a_j| = 1$ if i and j are neighbours. Supplemented with an appropriate Boltzmann weight, this is the integrable RSOS model of [6]. It is integrable in the sense that its Botzmann weights satisfy the Yang-Baxter equation and it possesses several critical regimes, in which its correlation length diverges. Since we are mainly interested in the connection with conformal theories, we shall concentrate on one of these critical domains. The constraints above have been reinterpreted in [7] as follows : heights are regarded as the vertices of the graph $\overset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\bullet}\cdots\cdots\overset{n}{\bullet}$, the second constraint expressing that neighbouring sites on the lattice are mapped onto neighbouring vertices on the graph. It is quite natural to try to generalize this picture to more general graphs. The search of critical and integrable models within this framework has to be supplemented by a certain Ansatz on the form of the Boltzmann weights, built out of representations of some algebra $\mathcal{T}$, typically some quotient of the Hecke algebra (a deformation of the algebra of the symmetric group) acting on the space of paths on the graph. The simplest case is given by the Temperley-Lieb algebra [8]. I shall not be more explicit on this point, referring the reader to the literature [9]. So within such an Ansatz, we have to face the

Problem 3 : *Find all the graphs that support a representation of the algebra $\mathcal{T}$ and hence lead to a critical integrable RSOS model.*

1.3. Topological Field Theories

In topological –better called cohomological– field theories (tft) one is dealing with a finite set of fields ϕ_i, $i = 1, \cdots, n$ living (in a two-dimensional case) on some Riemann surface Σ of genus g. These fields are in the cohomology space of a certain nilpotent operator Q, $Q\phi = 0$. One is interested in the correlation functions

$$\langle \phi_{i_1} \cdots \phi_{i_k} \rangle_\Sigma \ . \tag{1.10}$$

One assumes that all correlators between the ϕ's and a Q-exact field vanish. An example of such a Q-exact field is provided by the energy-momentum tensor that describes as usual

the response of the theory to a change of metric : $T = \{Q, G\}$. As a result of these axioms, the correlator (1.10) is insensitive to the representative of ϕ_i in its cohomology class and insensitive to a change of metric : it is a topological invariant that depends only on the labels $i_1, \cdots, i_k$ and on the genus g of Σ. In fact, one is considering *families* of such correlation functions, depending on n moduli $t_1, \cdots, t_n$ in one-to-one correspondence with the fields $\phi_1, \cdots, \phi_n$. (What is the actual topological meaning of this object remains to be seen and is a matter of a case by case analysis.)

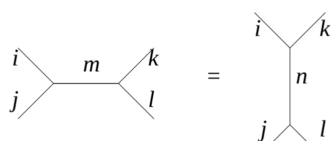
One then demands that these correlation functions satisfy a certain number of constraints:

(i) the 3-point genus 0 function $\langle \phi_i \phi_j \phi_k \rangle_0 = C_{ijk}(t_.)$ satisfies the integrability conditions that enable one to write it as

$$C_{ijk}(t_.) = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} \qquad \text{for some function } F(t_.) \ ; \tag{1.11}$$

(ii) the 2-point genus 0 function $\langle \phi_i \phi_j \rangle_0 = \eta_{ij} = C_{1ij}$ is an invertible, t-independent metric and may be used to raise and lower the indices of C_{ijk};

(iii) the 3-point functions satisfy the condition of factorization



$$\text{i.e.} \qquad \sum_m C_{ijm} C^m_{\ \ kl} = \sum_n C_{ikn} C^n_{\ \ jl} \ . \tag{1.12}$$

(iv) One usually adds the condition that $F(t_.)$ is a quasihomogeneous function of the t's. These constraints result in a set of non-linear partial differential equations for F, the so-called Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations [10–11].

Problem 4 : *Find solutions to these WDVV equations.*

As such, the problem is quite vast and it is remarkable that results may be obtained in such a general framework [12]. For our purpose, however, it is still too general and has to be supplemented with a certain Ansatz on the tft. In the following this will be provided by the connection with $N = 2$ theories obtained by the Kazama-Suzuki coset construction.

$N = 2$ superconformal theories are known to provide a class of tft's, after a modification called twisting. They are conformal theories endowed with a larger algebra, generated by the energy momentum tensor $T(z)$, two supersymmetry generators denoted $G^{\pm}(z)$ and a $U(1)$ current $J(z)$. I shall not write the algebra here, referring the reader to the literature [13]. Suffice it to say that the moment $G^+_{-\frac{1}{2}} = \oint \frac{dz}{2i\pi} G^+(z)$ is a nilpotent operator that we shall denote Q for brevity. The fields in the cohomology space of Q are called

chiral[1] ; their eigenvalues for L_0 and J_0 satisfy $h = \frac{1}{2}q$. The twisted energy-momentum $T^{\rm top}(z) = T(z) + \frac{1}{2}J(z)$ is Q-exact: $T^{\rm top}(z) = \frac{1}{2}\{Q, G^-(z)\}$ and the correlators of the chiral fields ϕ satisfy the axioms above. Deformations are introduced by perturbing the action by

$$\Delta S = \sum_{l=1}^{n} t_l \int \mathrm{d}^2 w\, G^-_{-\frac{1}{2}} \bar{G}^-_{-\frac{1}{2}} \phi_l(w,\bar{w}) + \text{c.c.} \tag{1.13}$$

and are shown to preserve the topological character of the correlators. Thus this procedure constructs a large class of solutions to the WDVV equations.

In turn, $N = 2$ theories may be constructed by a supersymmetric version of the coset construction of [3], due to Kazama and Suzuki. These theories are based on a pair (G, H) such that G is a simple compact Lie group, H a subgroup with rank $G =$ rank $H\ + 2n$ and $G/\left(H \times U(1)^{2n}\right)$ is a kählerian manifold [14]. (In the ordinary coset construction, they are described by the coset $(G \times SO(\dim G/H))/H$.) Those that will concern us are the grassmannians $SU(n{+}m)_k/\left(SU(n)_{k+m} \times SU(m)_{k+n} \times U(1)\right)$, where the label denotes the central charge of the affine algebra. Their Virasoro central charge is $c = 3nmk/(n{+}m{+}k)$.

2. The case of $SU(2)$

2.1. Classification of modular invariants

The classification of modular invariant partition functions has been completed for theories with a $\widehat{su}(2)$ affine algebra as a chiral algebra, or for some of their relatives through the coset construction. It is convenient to label the representations of $\widehat{su}(2)$ by the integer $\lambda = 2j + 1$, if j denotes the spin of the "horizontal" algebra. If the central charge is k, then $1 \le \lambda \le k + 1$. One finds after a fairly strenuous analysis that all modular invariant Z are in one-to-one correspondence with *simply laced* simple Lie algebras (of ADE type) [15]. Indeed if one singles out the diagonal terms in Z, writing

$$Z = \sum_{\lambda \in \{m_i\}} |\chi_\lambda|^2 + \text{non diagonal terms} \tag{2.1}$$

then one finds that the diagonal labels λ run over the *exponents* m_i of an ADE algebra of Coxeter number h related to the level k by $h = k + 2$. (Recall that the Coxeter number h and exponents m_j are integers $1 \le m_j \le h - 1$ such that the eigenvalues of the Cartan matrix C of an ADE algebra are $4 \sin^2 \frac{\pi m_i}{2h}$ or equivalently, those of the adjacency matrix of the Dynkin diagram $G = 2\mathbb{I} - C$ are $2 \cos \frac{\pi m_i}{h}$.)

[1] an unfortunate terminology, not to be confused with the concept of chiral algebra.

2.2. Determination of the structure constants

This program has also been carried through for $\widehat{su}(2)$ models and the corresponding "minimal" conformal theories [16]. The C_{IJK} are complicated, generically transcendant numbers, but somehow they also know about the ADE pattern ! This is apparent if, for two $\widehat{su}(2)$ theories with the same $h = k + 2$, one forms the *ratios* of structure constants pertaining to spin zero (left-right symmetric) fields. One finds that [17]

$$\frac{C^{(D)\text{ or }(E)}_{(ii)(jj)(\ell\ell)}}{C^{(A)}_{(ii)(jj)(\ell\ell)}} = \sum_a \frac{\psi_a^{(i)}\psi_a^{(j)}\psi_a^{(\ell)\,*}}{\psi_a^{(1)}} \ . \tag{2.2}$$

Here $\psi_a^{(i)}$ is the a-th component of the i-th orthonormalized eigenvector of the adjacency matrix of the D or E Dynkin diagram. The numbers that appear on the right hand side are algebraic (in fact they are square roots of rationals!) and form the structure constants of an associative and commutative algebra, first introduced in [18] in the context of lattice models. (For the A Dynkin diagram, this reduces to the Verlinde fusion algebra.)

Formula (2.2) does not seem to be a trivial consequence of the crossing equations (1.9). One has to go through the painstaking determination of all the C's to check it. On the other hand, as anticipated in [18], (2.2) is easily derived within the lattice models to be discussed now [17].

2.3. Height models on a graph

Although it may not be manifest, the RSOS models that have been introduced above have to do with the $SU(2)$ group. This may be seen through the chain of connections

XXZ $(SU(2))$ spin chain $\leftrightarrow$ 6 vertex model $\rightarrow$ RSOS model of ABF $\rightarrow$ new models

or at a more technical level, on the fact that the Temperley-Lieb algebra on which the construction relies is related to the $\mathcal{U}_q\, sl(2)$ quantum group [9]. In the continuum limit, the RSOS models are described by these minimal coset theories $SU(2)_k \times SU(2)_1/SU(2)_{k+1}$.

In his analysis of this class of height models attached to graphs, Pasquier showed that the condition of criticality is that the largest eigenvalue $\gamma_{\max}$ of the adjacency matrix be less or equal to 2. Leaving aside the case where it equals 2, which is a marginal case (related to conformal theories with $c = 1$), we recall the well known fact that the only graphs with $\gamma_{\max} < 2$ are the ADE Dynkin diagrams (plus the $\mathbb{Z}_2$ orbifold of the A_{2n} graph, that does not produce any new lattice model).

Hence in this context, we see that the ADE diagrams have been singled out by a spectral condition on their adjacency matrix.

2.4. Topological Field Theories

The simplest $N = 2$ "minimal" theories, constructed through the coset $(SU(2) \times SO(2))/U(1) \cong (SU(2)/U(1)) \times U(1)$, and the ensuing topological field theories, may be classified following the modular invariant procedure. One finds again an ADE classification, with some decoration associated to the $U(1)$ factors [19]. These decorations, however, do not affect the chiral content of these theories, and one finds [20] that all the chiral primary fields are spinless and have a $U(1)$ charge given by $\frac{m_i}{h}$, where h and m_i are as before the Coxeter number and exponents.

Another way of seing this ADE classification at work is through the Landau-Ginsburg approach. It has been shown that a class of $N = 2$ theories admits a description in terms of a superpotential. The latter is a function of some chiral superfields $X, Y, \cdots$ only, and has been argued to be a quasi-homogeneous polynomial $W(X, Y, \cdots)$ in these fields, with an isolated critical point at the origin: $W'_X|_0 = W'_Y|_0 = 0$. It is thus to be found in lists of singularities [21]. In particular, for the minimal theories, the superpotential must have no modulus and is among the well known ADE singularities [22]

$$\begin{array}{ll} A_n & W = \dfrac{X^{n+1}}{n+1} \\ D_{n+2} & W = \dfrac{X^{n+1}}{2(n+1)} + XY^2 \\ E_6 & W = \dfrac{X^3}{3} + \dfrac{Y^4}{4} \\ E_7 & W = \dfrac{X^3}{3} + \dfrac{XY^3}{3} \\ E_8 & W = \dfrac{X^3}{3} + \dfrac{Y^5}{5} \; . \end{array} \tag{2.3}$$

The $U(1)$ charges of the chiral fields, hence the gradings of the fields or their conjugate t parameters in the tft, are then read off the degrees of the local ring of the singularity; as is well known, they reproduce the Coxeter exponents of the ADE algebras [22].

We have thus found two independent routes to the ADE classification. The latter sheds some light on the former. Unfortunately it does not generalize easily to higher rank cases, due to the fact that many $N = 2$ theories escape the Landau-Ginsburg description.

This makes a third approach quite valuable. A year and a half ago, Dubrovin observed that one can associate a solution to the WDVV equations with each finite Coxeter group. Recall that Coxeter groups are generated by reflections in $(n-1)$-dimensional hyperplanes

in a n-dimensional Euclidean space V; the hyperplanes are orthogonal (through the origin) to a set of n independent vectors α_a called roots. Finite Coxeter groups are classified: beside the Weyl groups of the simple Lie algebras, A_p, B_p, C_p, (the two latter Coxeter groups being isomorphic), D_p, E_6, E_7, E_8, F_4 and G_2, there are the groups H_3 and H_4 of symmetries of the regular icosaedron and of a regular 4-dimensional polytope, and the infinite series $I_2(k)$ of the reflection groups of the regular k-gones in the plane. In general, if we use the roots α_a as a basis, in the reflection S_a in the hyperplane orthogonal to α_a, a vector $x = \sum_c x_c \alpha_c$ is transformed into

$$
\begin{aligned}
&S_a \; : \; x \mapsto x' = \sum x'_c \alpha_c = x - \frac{2(\alpha_a, x)}{(\alpha_a, \alpha_a)} \alpha_a \\
&\qquad\qquad \begin{cases} x'_a = -x_a + 2\sum_{c \neq a} G_{ac} x_c & \\ x'_b = x_b & \text{if } b \neq a \end{cases}
\end{aligned}
\tag{2.4}
$$

with $G = 2\mathbb{I} - C$ as in sect. 3.1. In Dubrovin's work, the homogeneity degrees of the variables t_i and of F are respectively $1 - (d_i - 2)/h$ and $2 + 2/h$ where h is the Coxeter number of the group G and d_i are the degrees of the G invariant polynomials in the coordinates of V. The ADE solutions are the previous minimal solutions. What are the other solutions, associated with non ADE finite Coxeter groups? They cannot be obtained by twisting of a consistent, modular invariant $N = 2$ theory. It seems that when coupled to gravity, only the ADE theories are consistent at higher genus [23].

Does the same pattern with all the Coxeter groups appear in the conformal context? The answer is yes, although the reason remains elusive. By inspection, one may check that the Coxeter groups label a certain class of subalgebras of the OPA of the ADE theories [24]. This class is defined as obeying two constraints: i) the subalgebra is generated by spinless (left-right symmetric) fields; ii) it contains the identity and the field of largest λ, i.e. $\lambda = h-1$, h the Coxeter number of the ADE algebra. Whether the latter constraint is merely technical or reflects something deeper remains to be seen. Note that again among all the theories endowed with these operator algebras, only the ADE ones are consistent (modular invariant) at non zero genus.

3. The case of $SU(N)$

3.1. Classification of $\widehat{su}(3)$ modular invariants

Let $\Lambda_1, \cdots, \Lambda_{N-1}$ be the fundamental weights of $su(N)$ and $\rho = \Lambda_1 + \cdots + \Lambda_{N-1}$ be their sum. Let $\mathcal{P}_{++}^{(k+N)}$ denote the set of integrable weights (shifted by ρ) of the affine algebra

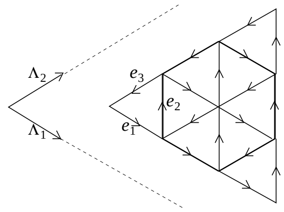


Fig. 1: $\mathcal{A}^{(6)} = \mathcal{P}_{++}^{(6)}$

$\widehat{su}(N)_k$ at level k, (the "Weyl alcove"),

$$\mathcal{P}_{++}^{(k+N)} = \{\lambda = \lambda_1\Lambda_1 + \cdots + \lambda_{N-1}\Lambda_{N-1} | \lambda_i \geq 1,\ \lambda_1 + \cdots \lambda_{N-1} \leq k + N - 1\} \ . \tag{3.1}$$

exemplified on $\widehat{su}(3)$ at level 3 in Fig. 1.

The partition function Z of the $\widehat{su}(N)_k$ theory must be a sesquilinear form in the characters labelled by some integrable weights

$$Z = \sum_\lambda |\chi_\lambda|^2 + \text{non diagonal terms} \tag{3.2}$$

and the representation $\lambda = \bar\lambda = \rho$ must appear once and only once (see (1.7)). In the case of $\widehat{su}(3)$, this is a problem with a fairly long history [25], that culminates with the recent proof by Gannon that the list of known invariants is complete [26]. The classification features again a diagonal series, several infinite series obtained by orbifolding or twisting of the former, and a few exceptional cases. This leaves us with the question : can one find a (geometrical or otherwise) meaning to this list ? By extension of the $su(2)$ case, we shall call *exponents* the weights λ that label the diagonal terms in Z and denote their set by Exp .

3.2. Construction of $SU(3)$ lattice models

The basic model – the analogue of the Andrews-Baxter-Forrester model – has heights λ living in $\mathcal{P}_{++}^{(k+N)}$, the set of integrable weights at a given level, and Boltzmann weights given by a representation of (a quotient of) the Hecke algebra [27–28]. This may be regarded as a model on the graph $\mathcal{A}^{(k+3)} = \mathcal{P}_{++}^{(k+N)}$ whose bonds are oriented along the three vectors of vanishing sum $e_1 = \Lambda_1$, $e_2 = \Lambda_2 - \Lambda_1$ and $e_3 = -\Lambda_2$ (see fig. 1). Its vertices may be 3-coloured, i.e. assigned a $\mathbb{Z}_3$ grading $\tau(.)$ (the "triality") in such a way that for the adjacency matrix G, $G_{ab} \neq 0$ only if $\tau(b) = \tau(a) + 1 \bmod 3$. This adjacency matrix has eigenvalues

$$\gamma^{(\lambda)} = \sum_{i=1}^{3} \exp \frac{2i\pi}{h}(e_i, \lambda) \ , \tag{3.3}$$

with λ running over the set $\mathcal{P}_{++}^{(k+N)}$, $(.,.)$ is the Cartan inner product in weight space and $h = k+3$. It is easy to see that the transposed matrix G^t that describes the graph with all orientations reversed commutes with G. Thus G is a normal matrix, that may be diagonalized in an orthonormal basis. Its diagonalization is indeed provided by the Verlinde formula, since G and G^t are the matrices of fusion by the two fundamental 3-dimensional representations of $SU(3)$.

Guided by the analogy with the case of $SU(2)$, one may look for graphs $\mathcal{G}$ satisfying the following properties:

i) $\mathcal{G}$ is 3-colourable, i.e. a $\mathbb{Z}_3$ grading τ may be assigned to its vertices, such that $G_{ab} \neq 0$ only if $\tau(b) = \tau(a) + 1 \bmod 3$;

ii) the graph is invariant under an involution $a \mapsto \bar{a}$ such that $\tau(\bar{a}) = -\tau(a)$ and $G_{\bar{a}\bar{b}} = G_{ba}$;

iii) its adjacency matrix is normal: $[G, G^t] = 0$;

iv) the spectrum of eigenvalues of G is a subset Exp (with possible multiplicities) of the set (3.3), for some h. In other words, this spectrum is described by a set of "exponents" $\lambda \in \mathcal{P}_{++}^{(h)}$ occuring with a multiplicity N_λ ;

v) these exponents match the exponents of one of the $SU(3)$ coset modular invariants; in particular, ρ belongs to the exponents, with multiplicity one.

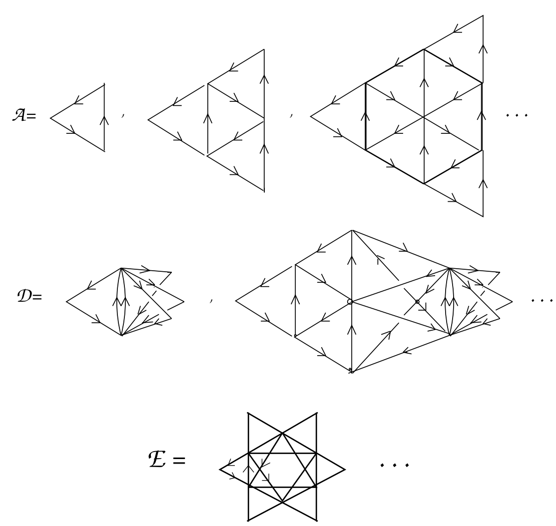


Fig. 2: Some of the $SU(3)$ graphs. (Not all orientations of edges have been shown on the last one.)

In [29] and [30], other solutions than the graphs $\mathcal{A}$ above were found; in fact we found enough solutions to match all the modular invariants (of coset theories) (see Fig.2). They come again in infinite series called $\mathcal{A}$ and $\mathcal{D}$ and a few exceptional cases $\mathcal{E}$; these notations are suggested by the analogy with the pattern of the previous case of $SU(2)$, but it should be stressed that there is no relation with the ordinary ADE scheme. We checked in almost all cases that these graphs support a representation of the appropriate Hecke algebra, hence an integrable $SU(3)$ height model. (Contrary to the case of $SU(2)$, this construction had to be done case by case and was more and more cumbersome as the complexity of the graph increased [30–31].) It is believed that the continuum limit of these lattice models reproduces the corresponding (coset) cft's. In [30], the construction of the graph corresponding to a given modular invariant was carried out in an empirical way, essentially by trials and errors. We now have some indications that there exists a direct procedure for determining the graph [32].

3.3. TFT's based on $SU(3)$

By the Kazama-Suzuki construction, one knows that the cosets

$$\frac{SU(3)_k}{SU(2)_{k+1} \times U(1)} \times SO(4)_1$$

lead to $N = 2$ theories and, after twisting, to topological field theories. As before for the minimal case, the classification of $SU(3)$ modular invariants reflects itself on that of these coset theories and the resulting tft's. A question is: what is the distinctive feature of tft's emanating from this series of cosets?

4. New Coxeter groups

With each graph $\mathcal{G}$ of the previous type, of adjacency matrix G_{ab}, we can associate a Coxeter group as follows. We consider a vector space V on $\mathbb{R}$, with a basis $\{\alpha_a\}$. We define a bilinear symmetric form on V by

$$\begin{aligned}(\alpha_a, \alpha_a) &= 1 \\ (\alpha_a, \alpha_b) &= \frac{1}{2}\,(G_{ab} + G_{ba}) \qquad \text{if} \quad a \neq b\ .\end{aligned} \tag{4.1}$$

This allows to define the reflexion S_a

$$S_a \quad : \quad x \mapsto x' = x - 2(\alpha_a, x)\alpha_a \tag{4.2}$$

or in terms of the components $x_b : x = \sum_b x_b \alpha_b$

$$\begin{aligned} x'_a &= -x_a - \sum_c (G_{ac} + G_{ca}) x_c \\ x'_b &= x_b \qquad \text{if} \quad b \neq a \ . \end{aligned} \tag{4.3}$$

Note the change of sign with respect to the case considered above (2.4).

Let us denote $\tilde{G}_{ab} = (G_{ab} + G_{ba})$. The following properties are standard or easily proved:

(i) $S_a^2 = \mathbf{1}$.

(ii) If $\tilde{G}_{ab} = 0$, then S_a and S_b commute and the product $S_a S_b$ is of order 2

$$(S_a S_b)^2 = \mathbf{1} \ .$$

(iii) If $\tilde{G}_{ab} = 1$, then $S_a S_b$ is of order 3 and more generally, if $\tilde{G}_{ab} = 2\cos\frac{\pi p}{q}$, with p and q coprime integers, $S_a S_b$ is of order q.

(iv) With h defined as in (3.3), the bilinear form $(\alpha_a, \alpha_b) = \delta_{ab} + \frac{1}{2}(G_{ab} + G_{ba})$ is positive definite for $h < 6$, semi–definite for $h = 6$ and indefinite for $h > 6$.

These properties are readily established. Less trivial is the following proposition, that extends to the present case the well-known relationship between the exponents of finite Coxeter groups and the eigenvalues of the "Coxeter element". Let

$$R = \prod_{\tau(a)=0} S_a \prod_{\tau(b)=1} S_b \prod_{\tau(c)=2} S_c \ , \tag{4.4}$$

obtained by the product of the three blocks of reflections of given triality. Then

Proposition : *The element R is independent, up to conjugation, of the order of the blocks and of the order of the S within each block, and its spectrum is of the form*

$$-\exp 3\frac{2i\pi}{h}(e_i, \lambda) \qquad \lambda \in \mathrm{Exp}\ \mathcal{G}, \quad i \text{ fixed} : 1 \leq i \leq 3 \tag{4.5}$$

with the same notations as in (3.3). In particular this set of eigenvalues is independent of $i = 1, 2, 3$, up to a permutation.

The proof relies on a simple extension of the original proof by Coxeter of the analogous statement for finite Coxeter groups. By a possible reordering of the vertices of the graph assume that the matrix G takes the block form

$$G = \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & B \\ C & 0 & 0 \end{pmatrix} \ . \tag{4.6}$$

Then the numbers $\epsilon_i^{(\lambda)} := \exp \frac{2i\pi}{h}(e_i, \lambda)$ are the roots of the polynomial

$$
\begin{aligned}
\prod_\lambda (z-\epsilon_1^{(\lambda)})(z-\epsilon_2^{(\lambda)})(z-\epsilon_3^{(\lambda)}) &= \det(z^3 - z^2 G + zG^t - \mathbf{1}) \\
&= \det \begin{pmatrix} (z^3-1)\mathbf{1} & -z^2 A & zC^t \\ zA^t & (z^3-1)\mathbf{1} & -z^2 B \\ -z^2 C & zB^t & (z^3-1)\mathbf{1} \end{pmatrix} = \Delta(z^3)
\end{aligned}
\tag{4.7}
$$

which is in fact a polynomial in z^3 as is readily seen by multiplying or dividing the rows and columns by the appropriate power of z. In fact

$$
\Delta(z^3) = \det(z^3 T - T^t) \tag{4.8}
$$

where

$$
T = \begin{pmatrix} \mathbf{1} & -A & C^t \\ 0 & \mathbf{1} & -B \\ 0 & 0 & \mathbf{1} \end{pmatrix} . \tag{4.9}
$$

Then it is easy to prove that the "Coxeter element" of (4.4) is conjugate to $-T^{-1}T^t$ and has thus by the previous discussion the spectrum $\{-\epsilon^{(\lambda)\,3}\}$. Finally, the independence up to conjugacy with respect to the order in (4.4) follows from the fact that each block in (4.4) has square one.

According to the point (iv) above, the groups generated by the S_a is finite for $h < 6$, and must therefore identify with one of the well known finite Coxeter groups. One verifies indeed that the groups associated with the graphs $\mathcal{A}^{(4)}$ (the $SU(3)$ weights of level 1), $\mathcal{A}^{(5)}$ (the same at level 2) and $\mathcal{A}^{(4)}$ in which all the links carry $G_{ab} = 2\cos\frac{\pi}{5}$ coincide respectively with the finite groups A_3, D_6 and H_3 of orders 24, $2^5 6!$ and 120. The first two identifications could have been anticipated from identifications between coset realizations of $N = 2$ theories. Indeed (see for instance [33])

$$
\begin{aligned}
\frac{SU(3)_1}{SU(2)_2 \times U(1)} \times SO(4)_1 &\equiv \frac{SU(2)_2}{U(1)} \times U(1) \\
\frac{SU(3)_2}{SU(2)_3 \times U(1)} \times SO(4)_1 &\equiv \Big[\frac{SU(2)_3}{U(1)} \times U(1)\Big]_{\text{“}D_6\text{”}} .
\end{aligned}
\tag{4.10}
$$

Thus we see that the group is more intrinsically attached to $N = 2$ theories or to their twisted, topological partner. This is very natural in view of the recent work of Dubrovin [12] who has shown that in tft's a major role is played by the monodromy group of a certain differential system. Under some assumptions, he was able to prove that this group is generated by reflections. For the minimal models discussed in sect.2.4, these groups

are the finite Coxeter groups. More generally it is suggested that the group that I just exhibited is this monodromy group. Beside the cases of minimal models quoted above, I have checked that the group that occurs in the case $\mathcal{A}^{(6)}$ is indeed the monodromy group computed in [12]. Also, the matrix T of (4.4) receives a natural interpretation in that context, as the Stokes matrix between two asymptotic behaviors of some function.

In contrast, if we return to the ordinary ("$N = 0$") cft's or integrable lattice models, it appears that the same group may be attached to two different models. These two different models, (pertaining for instance to $SU(2)$ and $SU(3)$, like in (4.10)), are in fact associated with two different *presentations* of the group; the generators (resp. the roots) may, depending on the case, be organised into two or three sets of commuting (resp. orthogonal) elements.

As in the case of $SU(2)$ above, it is expected that non integer values of G_{ab} describe certain subalgebras of the OPA. This is the case with the graph $\mathcal{A}^{(4)}$ with $G_{ab} = 2\cos\frac{\pi}{h}$ which describes the subalgebra of the $\mathcal{A}^{(h)}$ case generated by the left-right symmetric fields belonging to the orbit of the identity operator under the $\mathbb{Z}_3$ automorphism of the Weyl alcove, i.e. labelled by $\lambda = \bar{\lambda} \in \{(1,1),(1,h-2),(h-2,1)\}$.

Finally it should be noted that all these considerations extend nicely to general $SU(N)$. This will be expounded in a separate publication.

5. Conclusion

I have shown that with each of the graphs introduced in the construction of integrable lattice models one may associate in a natural way a Coxeter group. Conversely given a Coxeter group with a certain presentation by generators can one determine the graph? More precisely can one understand which class of groups occurs in connection with, say, $SU(N)$? How does the N colourability of the graph manifest itself? Can one classify the corresponding groups?

I have mentionned that within tft's, this group is the monodromy group introduced by Dubrovin. The most immediate question that arises is to find an interpretation of this Coxeter group in the context of ordinary cft's and/or lattice models. I have given hints that such an interpretation must be connected with the structure of the OPA, but at this stage this remains a vague and uncertain idea.

Clearly, there is ample room for more . . . reflections !

Acknowledgements I have benefited a lot from conversations with B. Dubrovin. It is also a pleasure to thank D. Bernard, C. Itzykson, V. Pasquier and V. Petkova for useful discussions.

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